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Note on the angular momentum of axi-symmetric isolated radiative systems

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Abstract. The Landau–Lifschitz pseudotensor is used to calculate the total angular momentum of an isolated radiative system in the full (axially symmetric) gravitational theory. The result differs from that derived from the Tamburino and Winicour definition of angular momentum, which is not based on a pseudotensor and which leads to a constant component of angular momentum about the symmetry axis. It is shown that the results cannot be brought into agreement by performing a BMS transformation.

1. Introduction

It is well known that in the case of a general isolated radiative system all definitions of the total energy–momentum 4-vector P_a lead to the same result, namely†

$$P_a(u) = \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} M(\delta_a^0 + Y_{1\alpha}\delta_a^\alpha)\sin\theta \, d\theta \, d\phi, \tag{1.1}$$

where $M(u, \theta, \phi)$ is the Bondi mass aspect (Bondi *et al* 1962) and $Y_{1\alpha}(\theta, \phi)$ are spherical harmonics. The uniqueness of the value of P_a is connected with the fact that, as in flat-space theories, the translations form a normal subgroup of the BMS group. This is in contradistinction to the case of angular momentum. In flat-space theories the angular momentum transforms inhomogeneously under a translation, but the inhomogeneous part $(x_{[a}P_{b]})$ has the correct tensorial transformation properties under a homogeneous Lorentz transformation. In the full theory one would like to have a quantity with similar transformation properties under translations and Lorentz transformations. However, the inhomogeneous term does not transform tensorially due to the mixing of momenta and supermomenta (Tamburino and Winicour 1966). Of the many definitions of angular momentum, we shall be concerned with two: one proposed by Landau and Lifschitz (1962) and the other by Tamburino and Winicour (1966), which we shall refer to subsequently as LL and TW, respectively.

Now Møller (1966) and Goldberg (1963), among others, have shown that the total energy–momentum 4-vector defined in terms of a pseudotensor also leads to the expression (1.1) for an isolated radiative system. Thus, since pseudotensor definitions and

† The notation is as follows: small Latin indices run from 0 to 3, Greek from 1 to 3, large Latin from 2 to 3. The signature is -2 and the Bondi coordinates are $(x^0, x^1, x^2, x^3) = (u, r, \theta, \phi)$. Derivatives with respect to u, r and θ are denoted by X_0, X_1 and X_2 respectively, although on some occasions commas are used in addition for clarity. Symmetrized and anti-symmetrized expressions do not include a numerical factor, eg $X_{(ab)} = X_{ab} + X_{ba}$.

other definitions both lead to the same result for the energy–momentum 4-vector, for radiative systems, the question arises: do pseudotensor and other definitions result in the same value for the angular momentum? In this paper we use the LL pseudotensor to compute an expression for the angular momentum and compare it with the equivalent expression derived from the TW definition. We conclude that the results do not agree and cannot be brought into agreement by performing a BMS transformation. We have chosen the LL pseudotensor since it alone satisfies certain criteria discussed by Goldberg (1958); in particular its symmetry leads to a natural definition of angular momentum. The choice of the TW definition rests on the general belief that it gives the 'correct' answer.

All calculations are carried out in the axi-symmetric case (see appendix 1) for which an invariant axis of rotation is defined. This also corresponds to the case most frequently considered in the literature. The formalism largely follows that of Bondi *et al* (1962). We choose as our metric the form of Sachs' metric (Sachs 1962) due to van der Burg (1966), the contravariant form of which is

$$\bar{g}^{ab} = \begin{pmatrix} 0 & e^{-2\beta} & 0 & 0 \\ \cdot & \dfrac{-V}{r}e^{-2\beta} & Ue^{-2\beta} & \dfrac{We^{-2\beta}}{\sin\theta} \\ \cdot & \cdot & \dfrac{-(\cosh 2\delta)\,e^{-2\gamma}}{r^2} & \dfrac{\sinh 2\delta}{r^2\sin\theta} \\ \cdot & \cdot & \cdot & \dfrac{-(\cosh 2\delta)\,e^{2\gamma}}{r^2\sin^2\theta} \end{pmatrix} \tag{1.2}$$

where in the axi-symmetric case β, γ, δ, U, V, W are all functions of u, r, θ only.

2. The Landau–Lifschitz definition

The LL energy–momentum complex τ^{ab} is given by

$$\tau^{ab} = (-g)(T^{ab}+t^{ab}) \tag{2.1}$$

where T^{ab} is the energy–momentum tensor and t^{ab} is the LL pseudotensor. In terms of the metric,

$$\tau^{ab} = \frac{1}{16\pi}[(-g)(g^{ab}g^{cd}-g^{ac}g^{bd})]_{,cd} \tag{2.2}$$

which is clearly symmetric, and by (2.1) defines t^{ab} *in vacuo*. If we set

$$p^{abc} = x^{[a}\tau^{b]c} \tag{2.3}$$

we obtain the conservation law

$$p^{abc}{}_{,c} = 0$$

and thus we can define the angular momentum ${}_{\rm LL}M^{ab}$ by

$${}_{\rm LL}M^{ab} = \int p^{abc}\,\mathrm{d}S_c \tag{2.4}$$

where, in the radiative case, the integration is over a null hypersurface. The integral can be converted using Stoke's theorem to an integral over Σ, a 2-surface on $\mathscr{I}^+$ (future null infinity).

3. The calculation

The quantity ${}_{\mathrm{LL}}M^{ab}$ was calculated in the radiative case by employing a method due to Møller (1966) which consists essentially of performing calculations in terms of a set of four quantities (μ_a, n_a, m_a, l_a) which asymptotically form a quasi-orthonormal tetrad of one null and three space-like vectors. In this section we use the following notation:

Bondi–Sachs coordinates	$(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3) = (u, r, \theta, \phi)$
Cartesian coordinates	$(x^0, x^1, x^2, x^3) = (t, x, y, z)$

where

$$(t, x, y, z) = (u+r, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta). \tag{3.1}$$

Then the tetrad vectors are defined to be

$$\mu_a = (1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta)$$

$$n_a = (0, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$m_a = (0, \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$l_a = (0, -\sin\phi, \cos\phi, 0).$$

Raising the indices with the Minkowski $\eta^{ab} = \mathrm{diag}(1, -1, -1, -1)$ gives rise to the orthonormality relations

$$n^a n_a = m^a m_a = l^a l_a = -n^a \mu_a = \mu^\alpha \mu_\alpha = n^\alpha n_\alpha = m^\alpha m_\alpha = l^\alpha l_\alpha = -n^\alpha \mu_\alpha = 1 \tag{3.2}$$

with all other products zero; from which we obtain the completeness relation

$$\eta_{ab} = \mu_a \mu_b + \mu_{(a} n_{b)} - m_a m_b - l_a l_b. \tag{3.3}$$

From the fact that the transformation matrices $\partial\bar{x}^a/\partial x^b$ and $\partial x^a/\partial\bar{x}^b$ can be expressed simply in terms of the tetrad vectors, we can now compute the Cartesian derivative of any function $\psi = \psi(u, r, \theta, \phi)$, by using

$$\psi_{,a} = \psi_{,\bar{0}}\mu_a + \psi_{,\bar{1}} n_a + \psi_{,\bar{2}} \frac{m_a}{r} + \psi_{,\bar{3}} \frac{l_a}{r\sin\theta}. \tag{3.4}$$

Defining the quantities

$$16\pi\lambda^{abcd} = [(-g)(g^{ab}g^{cd} - g^{ac}g^{bd})] = \mathfrak{g}^{ab}\mathfrak{g}^{cd} - \mathfrak{g}^{ac}\mathfrak{g}^{bd} \tag{3.5}$$

where $\mathfrak{g}^{ab}$ is the metric density, and

$$h^{abc} = \lambda^{abcd}{}_{,d} \tag{3.6}$$

it is straightforward to show that

$$x^{[a}\tau^{b]c} = (x^{[a}h^{b]cd} + \lambda^{abcd})_{,d}. \tag{3.7}$$

Then applying Stoke's theorem to (2.4) we find

$${}_{\mathrm{LL}}M^{ab} = \tfrac{1}{2}\oint_{\partial V} (x^{[a}h^{b]cd} + \lambda^{abcd})\, \mathrm{d}S^*_{cd} \tag{3.8}$$

where

$$\mathrm{d}S^*_{ab} = \tfrac{1}{2}\epsilon_{abcd}\,\mathrm{d}S^{cd}$$

is the dual of $\mathrm{d}S^{ab}$. Taking ∂V to be Σ, (3.8) becomes

$$_{\mathrm{LL}}M^{ab} = \lim_{a\to\infty}\int_0^{\pi}\int_0^{2\pi}(x^{[a}h^{b]0\alpha}+\lambda^{a0\alpha b})n_\alpha a^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{3.9}$$

The evaluation of this expression is facilitated by computing the two expressions

$$\lambda^{a0\alpha b}n_\alpha \tag{3.10}$$

and

$$h^{a0\alpha}n_\alpha = (\lambda^{a0\alpha b}n_\alpha)_{,b}-\lambda^{a0\alpha b}(n_\alpha)_{,b} \tag{3.11}$$

where the contraction with n_α considerably reduces the number of terms occurring. The required parts of the metric density of the metric (1.2) are

$$\mathfrak{g}^{0a} = \delta^a_0+(1-Vr^{-1})\mu^a-Urm^a-Wrl^a \tag{3.12}$$

and

$$\mathfrak{g}^{\alpha a} = \mu^\alpha\delta^a_0-Vr^{-1}\mu^\alpha\mu^a-Ur\mu^{(\alpha}m^{a)}-Wr\mu^{(\alpha}l^{a)}-\cosh 2\delta\,\mathrm{e}^{-2\gamma+2\beta}m^\alpha m^a$$
$$-\cosh 2\delta\,\mathrm{e}^{2\gamma+2\beta}l^\alpha l^a+\sinh 2\delta\,\mathrm{e}^{2\beta}m^{(\alpha}l^{a)}. \tag{3.13}$$

Inserting these expressions into (3.9) and making repeated use of the orthonormality relations (3.2) and equation (3.3) we find that the axial symmetry leads directly to

$$_{\mathrm{LL}}M^{01} = {}_{\mathrm{LL}}M^{02} = {}_{\mathrm{LL}}M^{03} = {}_{\mathrm{LL}}M^{23} = 0. \tag{3.14}$$

The remaining components involve the asymptotic expansions of the arbitrary functions in the metric (1.2) (see appendix 2). A final requirement is that the metric be regular on the axis of symmetry. Two of the conditions for this are that $\gamma\sin^{-2}\theta$ and $\delta\sin^{-2}\theta$ are regular functions of $\cos\theta$ as $\sin\theta\to 0$. This implies the following limiting behaviour on the news functions c and d:

$$\left.\begin{aligned} c &\simeq k(u)\sin^2\theta \\ d &\simeq j(u)\sin^2\theta \end{aligned}\right\} \qquad \text{as } \sin\theta\to 0. \tag{3.15}$$

These conditions are called the regularity conditions. A lengthy calculation eventually leads to the expressions

$$_{\mathrm{LL}}M^{03} = uP^3-\tfrac{1}{4}\int_0^{\pi}[3N-2(cc_2+dd_2)]\sin^2\theta\,\mathrm{d}\theta, \tag{3.16}$$

and

$$_{\mathrm{LL}}M^{12} = \tfrac{1}{4}\int_0^{\pi}[3P+2(c_2d-cd_2)]\sin^2\theta\,\mathrm{d}\theta. \tag{3.17}$$

4. The Tamburino–Winicour definition

We now turn to the angular momentum definition of TW, who start by defining a flux linkage

$$L_\xi(\Sigma) = \int_\Sigma(\xi^{[a;b]}-\xi^c{}_{;c}k^{[a}m^{b]})\,\mathrm{d}S_{ab} \tag{4.1}$$

which they arrive at by modifying the integral form of a covariant flux conservation law due to Komar in which the second term in the above integrand is absent. In particular if ξ^a is a global Killing field the above definition reduces to Komar's and leads to a conservation law. However, in general, a space–time does not possess a global Killing field, but if the system is isolated the space–time will possess an asymptotic Killing field. In the radiative case the asymptotic symmetry group is the BMS group. TW use the flux linkage to define the total angular momentum ${}_{\mathrm{TW}}M_{ab}$ in terms of a conformal Bondi metric g_{ab}, related to the physical metric $\bar{g}_{ab}$ of (1.2) by

$$g_{ab} = r^2 \bar{g}_{ab}, \tag{4.2}$$

and also in terms of the descriptors ξ^a of the asymptotic symmetry group. They obtain the expression

$${}_{\mathrm{TW}}M^{ab} = \oint_{\Sigma} [\tfrac{1}{2}(M - \tfrac{1}{4}g^{1A}{}_{,11A})\xi^{A}_{[ab]:A}u + (g_{AB}N^B - \tfrac{1}{4}g_{01,11A} - \tfrac{1}{2}g_{AB,1}g^{1B}{}_{,11})\xi^{A}_{[ab]}]\,\mathrm{d}S \tag{4.3}$$

where the colon represents two-dimensional covariant differentiation with respect to the polar metric g_{AB} and where

$$N^A(u, \theta, \phi) = \lim_{r \to 0} (-\tfrac{1}{4}g^{1A}{}_{,111}). \tag{4.4}$$

In the axi-symmetric case we have a global Killing field and the descriptor corresponding to this symmetry generates the component

$${}_{\mathrm{TW}}M_{12} = \tfrac{1}{4}\int_0^{\pi} [3P + (c_2 d - c d_2)] \sin^2\theta \,\mathrm{d}\theta. \tag{4.5}$$

In addition the descriptor corresponding to a boost along the symmetry axis generates the component

$${}_{\mathrm{TW}}M_{03} = uP_3 + \tfrac{1}{4}\int_0^{\pi} 3N \sin^2\theta \,\mathrm{d}\theta. \tag{4.6}$$

All other components of ${}_{\mathrm{TW}}M_{ab}$ are zero.

If we now use the supplementary conditions (see appendix 3) and the regularity conditions we readily verify that the retarded time derivative of (4.5)

$$\frac{\mathrm{d}}{\mathrm{d}u}({}_{\mathrm{TW}}M_{12}) = 0 \tag{4.7}$$

so that ${}_{\mathrm{TW}}M_{12}$ is, in fact, a constant.

5. Comparison of results

Collecting the results of the last two sections together, the only non-vanishing component of the space-like part of the total angular momentum in the axi-symmetric case is the component about the axis of symmetry, which in each case is

$${}_{\mathrm{LL}}M^{12} = \text{constant} + \tfrac{1}{4}\int_0^{\pi} (c_2 d - d_2 c) \sin^2\theta \,\mathrm{d}\theta \tag{5.1}$$

$${}_{\mathrm{TW}}M^{12} = \text{constant} \tag{5.2}$$

where the constant is

$$\tfrac{1}{4}\int_0^\pi [3P+(c_2d-d_2c)]\sin^2\theta\,\mathrm{d}\theta. \tag{5.3}$$

Thus the space-like part of the angular momentum is conserved in the TW case, but changes in the LL case unless the quantity

$$I = \int_0^\pi (c_2d-d_2c)\sin^2\theta\,\mathrm{d}\theta \tag{5.4}$$

is a constant. This will be a constant in particular if either the news (c_0+id_0) vanishes, or c or d vanishes, or $c = d$.

The TW definition would appear to be the more physically reasonable and is in agreement with the analogous result in flat space theories, namely that axial symmetry leads to a constant component of angular momentum about the symmetry axis. However, it might be thought possible to remove the extra term I in the LL result by a BMS transformation. We shall attempt to show that this is not possible. In passing we might add that the definition of angular momentum based on multiple moments proposed by Newman and Unti (1965) also leads to a result differing from the constant (5.3) by precisely the term I.

We start by considering a BMS transformation which, in the case of axial symmetry, (preserving the ϕ independence) is given on $\mathscr{I}^+$ by

$$u = K^{-1}\bar{u}+\alpha(\bar{\theta})$$

$$\theta = 2\tan^{-1}(\mathrm{e}^{-\nu}\tan\tfrac{1}{2}\bar{\theta})$$

$$\phi = \bar{\phi}+\mu$$

where $\mu =$ constant, $\nu =$ constant, $K = \cosh\nu+\cos\bar{\theta}\sinh\nu$, and $\alpha = \alpha(\bar{\theta})$. The functions c and d transform under such a transformation according to

$$\bar{c} = \frac{c}{K}+\tfrac{1}{2}K\left[\alpha'\left(\cot\bar{\theta}-\frac{2K'}{K}\right)-\alpha''\right]$$

and

$$\bar{d} = \frac{d}{K}.$$

We wish to prove that for all choices of $c(u,\theta)$ and $d(u,\theta)$ subject to the regularity conditions (3.15) there is *no* BMS transformation for which, in the new coordinate system,

$$\bar{I}_{,\bar{0}} = 0.$$

We shall do this by constructing explicit counter examples. Since any BMS transformation can be uniquely decomposed into a homogeneous Lorentz transformation and a supertranslation (in a definite order), we need only consider the transformation properties of $I_{,0}$ under the action of each subgroup separately.

5.1. Homogeneous Lorentz transformation: $\mu = \alpha = 0$, $\nu \neq 0$

Then

$$\bar{I}_{,\bar{0}} = \frac{\partial}{\partial u}\int_0^\pi (c_2d-d_2c)\frac{\sin^2\theta}{K}\mathrm{d}\theta+\frac{\partial}{\partial u}\int_0^\pi [(cd_0-dc_0)u]\frac{\sin^2\theta K'}{K}\mathrm{d}\theta. \tag{5.5}$$

If we choose

$$c = e^{pu}\sin^2\theta, \qquad d = e^{qu}\sin^2\theta \qquad (p \neq q)$$

where p and q are constants, then the coefficient of u in (5.5) is

$$\int_0^\pi \frac{\sin^6\theta}{K}\,d\theta. \tag{5.6}$$

Since

$$K = \tfrac{1}{2}e^{\nu}(1+\cos\bar{\theta}) + \tfrac{1}{2}e^{-\nu}(1-\cos\bar{\theta})$$

is clearly positive, then so is the integrand of (5.6) and hence the coefficient of u does not vanish for the given choice of c and d; in which case neither does $\bar{I}_{,\bar{0}}$. Of course the rotation $\alpha = \nu = 0, \mu \neq 0$ leaves $I_{,0}$ invariant.

5.2. Supertranslation: $\mu = \nu = 0, \alpha \neq 0$

Then

$$\bar{I}_{,\bar{0}} = \frac{\partial}{\partial u}\int_0^\pi (c_2 d - d_2 c)\sin^2\theta\,d\theta - \frac{\partial}{\partial u}\int_0^\pi [(cd_0 - dc_0)\alpha']\sin^2\theta\,d\theta$$
$$+\tfrac{1}{2}\int_0^\pi [(\alpha''\cot\bar{\theta} - \alpha'\operatorname{cosec}^2\bar{\theta} - \alpha''')d_0 - (\alpha' d_{00} + d_{02})$$
$$\times(\alpha'\cot\bar{\theta} - \alpha'')]\sin^2\theta\,d\theta. \tag{5.7}$$

If we choose

$$c = e^{pu}\sin^2\theta, \qquad d = e^{pu}\cos\theta\sin^2\theta$$

where p is a constant, then the coefficient of e^{2pu} in (5.7) is

$$\int_0^\pi \sin^7\theta\,d\theta$$

which is clearly positive, and so again $\bar{I}_{,\bar{0}}$ does not vanish. Thus the non-equivalence of (5.1) and (5.2) has been demonstrated.

6. Two important subcases

If we impose the additional condition of azimuth reflection invariance (see appendix 1) then the resulting metric is Bondi's metric (Bondi *et al* 1962). This reduction is accomplished by setting W and δ and hence in particular d and P to zero. Equations (5.1) and (5.3) then immediately lead to the vanishing of the space-like part of the angular momentum, as we should expect, since this condition corresponds intuitively to a non-rotating source.

Another important subcase, not often considered in the literature, is that of equatorial reflection invariance (see appendix 1). This condition leads to the following requirements:

$$
\begin{aligned}
M(u, \theta) &= M(u, \pi-\theta) \\
c(u, \theta) &= c(u, \pi-\theta) \\
d(u, \theta) &= -d(u, \pi-\theta) \\
N(u, \theta) &= -N(u, \pi-\theta) \\
P(u, \theta) &= P(u, \pi-\theta).
\end{aligned}
$$

Inserting these conditions into (1.1), (3.16) and (4.6) we find that the total momentum (P^3), momentum recoil ($P^3{}_{,0}$) and the non-space-like component of the angular momentum (${}_{\mathrm{LL}}M^{03}$ or ${}_{\mathrm{TW}}M^{03}$) all vanish. This is again what we might expect intuitively since equatorial reflection invariance suggests there should be no net motion along the symmetry axis and the 'centre of mass' of the system should appear to a distant observer to be at the coordinate origin (in as far as this makes sense). Of course, under the same conditions, the mass and angular momentum are not zero in general. We mention that perhaps the simplest physically interesting case of a radiating system is that in which the external field is axi-symmetric and possesses both azimuth and equatorial reflection symmetry (intuitively we might think of a non-rotating pulsating ellipsoid). In this case all quantities vanish except the mass (P^0) and the mass-loss ($P^0{}_{,0}$).

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Appendix 1. Symmetry definitions

We start by tying down polar coordinates $(x^2, x^3) = (\theta, \phi)$ on Σ in a standard manner and then extending to a Bondi coordinate system (u, r, θ, ϕ). Then using this coordinate system a space–time possesses:

(i) *axi-symmetry* if $\xi^a = \delta^a_3$ is a Killing vector field, ie generates an infinitesimal isometry. In coordinate terms this requires

$$g_{ab,3} = 0.$$

Such a metric defines invariantly an axis of symmetry which consists of those points left (point-wise) fixed under the action of the infinitesimal isometry. (An axi-symmetric space–time can be defined in a coordinate independent manner as one admitting a space-like Killing vector field whose orbits are topological circles).

(ii) *azimuth reflection invariance* if in addition to (i) the transformation

$$
\begin{aligned}
x^\alpha &\to x'^\alpha = x^\alpha \\
x^3 &\to x'^3 = -x^3
\end{aligned}
$$

is an isometry. In coordinate terms this requires

$$g_{03} = g_{13} = g_{23} = 0.$$

(In coordinate-independent terms the space–time in addition to (i) admits a (non-trivial) isometric action of Z_2 leaving a hyperplane containing the axis of symmetry pointwise invariant.)

(iii) *equatorial reflection invariance* if in addition to (i) the transformation

$$\begin{aligned} x^0 &\to x'^0 = x^0 \\ x^1 &\to x'^1 = x^1 \\ x^2 &\to x'^2 = \pi - x^2 \\ x^3 &\to x'^3 = x^3 \end{aligned}$$

is an isometry, ie

$$g_{ab}(x') = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd}(x).$$

(The space–time in addition to (i) admits a non-trivial isometric action of Z_2 leaving a hyperplane normal to the axis of symmetry pointwise invariant.)

Appendix 2. Asymptotic expansions

The asymptotic expansions of the arbitrary functions occurring in the metric (1.2), to the required order, are (in the axi-symmetric case)

$$\begin{aligned}
\gamma &= c(u,\theta)r^{-1} + O(r^{-3}) \\
\delta &= d(u,\theta)r^{-1} + O(r^{-3}) \\
\beta &= -\tfrac{1}{4}(c^2+d^2)r^{-2} + O(r^{-4}) \\
V &= r - 2M(u,\theta) + O(r^{-1}) \\
U &= -(c_2 + 2c\cot\theta)r^{-2} + [2N(u,\theta) + 3(cc_2 + dd_2) + 4(c^2+d^2)\cot\theta]r^{-3} + O(r^{-4}) \\
W &= -(d_2 + 2d\cot\theta)r^{-2} + [2P(u,\theta) + 2(c_2 d - cd_2)]r^{-3} + O(r^{-4}).
\end{aligned}$$

Appendix 3. Supplementary conditions

These are (in the axi-symmetric case)

$$\begin{aligned}
M_0 &= -(c_0^2 + d_0^2) + \tfrac{1}{2}v_0 \\
3N_0 &= -M_2 - (c_0c_2 + d_0d_2) - 3(cc_{02} + dd_{02}) - 4(cc_0 + dd_0)\cot\theta \\
3P_0 &= \tfrac{1}{2}\lambda_2 + (c_2d_0 - c_0d_2) + 3(cd_{02} - c_{02}d) + 4(cd_0 - c_0d)\cot\theta
\end{aligned}$$

where

$$v = \left(\frac{\partial}{\partial\theta} + \cot\theta\right)(c_2 + 2c\cot\theta)$$

and

$$\lambda = \left(\frac{\partial}{\partial\theta} + \cot\theta\right)(d_2 + 2d\cot\theta).$$

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